U.G. 6th Semester Examination - 2023

MATHEMATICS

[HONOURS]

Course Code : MATH-H-CC-T-13

(Metric Space and Complex Analysis)

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

The symbols and notations have their usual meanings.

1. Answer any **ten** questions: $2\times10=20$

 a) Prove or disprove: In a metric space (X, d), if $\lim_{n\to\infty} d(x_n, x_{n+1})=0$ then (x_n) is a Cauchy sequence in X.

 b) Given that G is an open subset of a metric space (X, d). Show that G is a union of open balls in (X, d).

 c) Prove or disprove: Continuous image of a locally connected metric space is locally connected.

[Turn Over]

d) Show that the set $X = \mathbb{R}$ with the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is bounded.

e) Show that the union of a finite number of closed sets in a metric space is closed.

f) In a metric space, show that the closure of a connected set is connected.

g) Show that a closed subset of a compact metric space is compact.

h) Show that $f(z) = \frac{1}{z^2}$ is uniformly continuous in $\frac{1}{2} \le |z| \le 1$.

i) Let $f = u + iv$ be analytic in a domain D. If Re f is constant in D, then show that f is constant in D.

j) Show that the function $f(z) = \frac{\overline{z}}{z}$ does not have a limit as $z \to 0$.

k) Prove that $f(z) = \text{Im } z$ is not differentiable at any point.

l) Show that $\int_C f(z)\, dz = 1 - i$, where $f(z) = y - x - 3ix^2$ and C is the line segment from $z = 0$ to $z = 1 + i$.

m) Evaluate $\oint_{|z|=2} \frac{z\,dz}{(9-z^2)(z+i)}$.

n) Expand $f(z) = \frac{z-1}{z+1}$ in a Taylor's series about the point $z = 0$.

o) If z is a complex number such that $z\overline{z} = 1$ then find the value of $|1+z|^2 + |1-z|^2$.

2. Answer any **four** questions: 5×4=20

a) Show that a compact metric space is always second countable.

b) Show that the metric space l_p $(1 < p < \infty)$ consisting of all real sequences $x = (x_1, x_2, \ldots)$ with $\left(\sum_{i=1}^{\infty} x_i^p\right)^{\frac{1}{p}} < \infty$, is complete with respect to the metric $d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}$ for $x, y \in l_p$.

c) State and prove Cantor's intersection theorem.

d) Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for

i) $1<|z|<3$

ii) $|z|>3$

e) Let $f(z)=\sqrt{|xy|}$. Show that though C-R equations are satisfied at origin but $f'(0)$ does not exist.

f) Let f be analytic in a simply connected region R and let α and β be any two points in R. Show that $\int_{\alpha}^{\beta} f(z)\,dz$ is independent of the path in R joining α and β.

3. Answer any **two** questions: 10×2=20

a) i) Prove that if f is one-one and onto continuous mapping of a compact metric space (X, d) into a metric space (Y, ρ) then f^{-1} is continuous on (Y, ρ).

ii) Using Cauchy's integral formula, evaluate

$$\int_C \frac{e^z}{(z+1)^2}\,dz$$ where C is the circle $|z-1|=3$. 5+5

b) i) If $G_1, G_2, \ldots, G_n$ are compact sets in a metric space (X, d) then show that $\bigcup_{i=1}^{n} G_i$

is a compact set of (X, d). Can you extend the result over an infinite number of such set in (X, d)? Give reason.

ii) If f(z) is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$ 5+5

c) i) Show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a convergent sequence in real number space with usual metric, and hence obtain $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$.

ii) Determine the region of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3\,4^n}.$$ 5+5